

# Longest Alternating Subsequences of Permutations<sup>1</sup>

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## Abstract

The length  $\text{is}(w)$  of the longest increasing subsequence of a permutation  $w$  in the symmetric group  $\mathfrak{S}_n$  has been the object of much investigation. We develop comparable results for the length  $\text{as}(w)$  of the longest alternating subsequence of  $w$ , where a sequence  $a, b, c, d, \dots$  is *alternating* if  $a > b < c > d < \dots$ . For instance, the expected value (mean) of  $\text{as}(w)$  for  $w \in \mathfrak{S}_n$  is exactly  $(4n + 1)/6$  if  $n \geq 2$ .

## 1 Introduction.

Let  $\mathfrak{S}_n$  denote the symmetric group of permutations of  $1, 2, \dots, n$ , and let  $w = w_1 \cdots w_n \in \mathfrak{S}_n$ . An *increasing subsequence* of  $w$  of length  $k$  is a subsequence  $w_{i_1} \cdots w_{i_k}$  satisfying

$$w_{i_1} < w_{i_2} < \cdots < w_{i_k}.$$

There has been much recent work on the length  $\text{is}_n(w)$  of the longest increasing subsequence of a permutation  $w \in \mathfrak{S}_n$ . A highlight is the asymptotic determination of the expectation  $E(n)$  of  $\text{is}_n$  by Logan-Shepp [11] and Vershik-Kerov [18], viz.,

$$E(n) := \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \text{is}_n(w) \sim 2\sqrt{n}, \quad n \rightarrow \infty. \quad (1)$$

Baik, Deift and Johansson [3] obtained a vast strengthening of this result, viz., the limiting distribution of  $\text{is}_n(w)$  as  $n \rightarrow \infty$ . Namely,

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for  $w$  chosen uniformly from  $\mathfrak{S}_n$  we have

$$\lim_{n \rightarrow \infty} \text{Prob} \left( \frac{\text{is}_n(w) - 2\sqrt{n}}{n^{1/6}} \leq t \right) = F(t), \quad (2)$$

where  $F(t)$  is the Tracy-Widom distribution. The proof uses a result of Gessel [9] that gives a generating function for the quantity

$$u_k(n) = \#\{w \in \mathfrak{S}_n : \text{is}(w) \leq k\}.$$

Namely, define

$$U_k(x) = \sum_{n \geq 0} u_k(n) \frac{x^{2n}}{n!^2}, \quad k \geq 1$$

$$I_i(2x) = \sum_{n \geq 0} \frac{x^{2n+i}}{n!(n+i)!}, \quad i \in \mathbb{Z}.$$

The function  $I_i$  is the *hyperbolic Bessel function* of the first kind of order  $i$ . Note that  $I_i(2x) = I_{-i}(2x)$ . Gessel then showed that

$$U_k(x) = \det (I_{i-j}(2x))_{i,j=1}^k.$$

In this paper we will develop an analogous theory for *alternating subsequences*, i.e., subsequences  $w_{i_1} \cdots w_{i_k}$  of  $w$  satisfying

$$w_{i_1} > w_{i_2} < w_{i_3} > w_{i_4} < \cdots w_{i_k}.$$

Note that according to our definition, an alternating sequence  $a, b, c, \dots$  (of length at least two) must begin with a descent  $a > b$ . Let  $\text{as}(w) = \text{as}_n(w)$  denote the length (number of terms) of the longest alternating subsequence of  $w \in \mathfrak{S}_n$ , and let

$$a_k(n) = \#\{w \in \mathfrak{S}_n : \text{as}(w) = k\}.$$

For instance,  $a_1(w) = 1$ , corresponding to the permutation  $12 \cdots n$ , while  $a_n(n)$  is the total number of alternating permutations in  $\mathfrak{S}_n$ . This number is customarily denoted  $E_n$ . A celebrated result of André [1][16, §3.16] states that

$$\sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x. \quad (3)$$

The numbers  $E_n$  were first considered by Euler (using (3) as their definition) and are known as *Euler numbers*. Because of (3)  $E_{2n}$  is also known as a *secant number* and  $E_{2n-1}$  as a *tangent number*.

Define

$$\begin{aligned} b_k(n) &= \#\{w \in \mathfrak{S}_n : \text{as}(w) \leq k\} \\ &= a_1(n) + a_2(n) + \cdots + a_k(n), \end{aligned} \quad (4)$$

so for instance  $b_k(n) = n!$  for  $k \geq n$ . Also define the generating functions

$$\begin{aligned} A(x, t) &= \sum_{k, n \geq 0} a_k(n) t^k \frac{x^n}{n!} \\ B(x, t) &= \sum_{k, n \geq 0} b_k(n) t^k \frac{x^n}{n!}. \end{aligned} \quad (5)$$

Our main result (Theorem 2.3) is the formulas

$$B(x, t) = \frac{1 + \rho + 2te^{\rho x} + (1 - \rho)e^{2\rho x}}{1 + \rho - t^2 + (1 - \rho - t^2)e^{2\rho x}} \quad (6)$$

$$A(x, t) = (1 - t)B(x, t),$$

where  $\rho = \sqrt{1 - t^2}$ .

As a consequence of these formulas we obtain explicit formulas for  $a_k(n)$  and  $b_k(n)$ :

$$\begin{aligned} b_k(n) &= \frac{1}{2^{k-1}} \sum_{\substack{r+2s \leq k \\ r \equiv k \pmod{2}}} (-2)^s \binom{k-s}{(k+r)/2} \binom{n}{s} r^n \\ a_k(n) &= b_k(n) - b_{k-1}(n). \end{aligned}$$

We also obtain from equation (6) formulas for the factorial moments

$$\nu_k(n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \text{as}(w)(\text{as}(w) - 1) \cdots (\text{as}(w) - k + 1).$$

For instance, the mean  $\nu_1(n)$  and variance  $\text{var}(\text{as}_n) = \nu_2(n) + \nu_1(n) - \nu_1(n)^2$  are given by

$$\begin{aligned} \nu_1(n) &= \frac{4n+1}{6}, \quad n \geq 2 \\ \text{var}(\text{as}_n) &= \frac{8}{45}n - \frac{13}{180}, \quad n \geq 4. \end{aligned} \tag{7}$$

The limiting distribution of  $\text{as}_n$  (the analogue of equation 2)) was obtained independently by Pemantle and Widom, as discussed at the end of Section 3. Rather than the Tracy-Widom distribution as in (2), this time we obtain a Gaussian distribution.

NOTE. We can give an alternative description of  $b_k(n)$  in terms of pattern avoidance. If  $v = v_1v_2 \cdots v_k \in \mathfrak{S}_k$ , then we say that a permutation  $w = w_1w_2 \cdots w_n \in \mathfrak{S}_n$  *avoids*  $v$  if  $w$  has no subsequence  $w_{i_1}w_{i_2} \cdots w_{i_k}$  whose terms are in the same relative order as  $v$  [6, Ch. 4.5][17, §7]. If  $X \subset \mathfrak{S}_k$ , then we say that  $w \in \mathfrak{S}_n$  *avoids*  $X$  if  $w$  avoids all  $v \in X$ . Now note that  $b_{k-1}(n)$  is the number of permutations  $w \in \mathfrak{S}_n$  that avoid all  $E_k$  alternating permutations in  $\mathfrak{S}_k$ .

After seeing the first draft of this paper Miklós Bóna pointed out that the statistic  $\text{as}_n$  can be expressed very simply in terms of a previously considered statistic on  $\mathfrak{S}_n$ , viz., the number of *alternating runs*. Hence many of our results can also be deduced from known results on alternating runs. This development is discussed further in Section 4. In particular, it follows from [20] that the polynomials  $T_n(t) = \sum_k a_k(n)t^k$  have interlacing real zeros. This result can be used to give a third proof (in addition to the proofs of Pemantle and Widom) that the limiting distribution of  $\text{as}_n$  is Gaussian.

## 2 The main generating function.

The key result that allows us to obtain explicit formulas is the following lemma.

**Lemma 2.1.** *Let  $w \in \mathfrak{S}_n$ . Then there is an alternating subsequence of  $w$  of maximum length that contains  $n$ .*

*Proof.* Let  $a_1 > a_2 < \cdots a_k$  be an alternating subsequence of  $w$  of maximum length  $k = \text{as}(w)$ , and suppose that  $n$  is not a term of this subsequence. If  $n$  precedes  $a_1$  in  $w$ , then we can replace  $a_1$  by  $n$  and obtain an alternating subsequence of length  $k$  containing  $n$ . If  $n$  appears between  $a_i$  and  $a_{i+1}$  in  $w$ , then we can similarly replace the larger of  $a_i$  and  $a_{i+1}$  by  $n$ . Finally, suppose that  $n$  appears to the right of  $a_k$ . If  $k$  is even that we can append  $n$  to the end of the subsequence to obtain a longer alternating subsequence, contradicting the definition of  $k$ . But if  $k$  is odd, then we can replace  $a_k$  by  $n$ , again obtaining an alternating subsequence of length  $k$  containing  $n$ .  $\square$

We can use Lemma 2.1 to obtain a recurrence for  $a_k(n)$ , beginning with the initial condition  $a_0(0) = 1$ .

**Lemma 2.2.** *Let  $1 \leq k \leq n + 1$ . Then*

$$a_k(n+1) = \sum_{j=0}^n \binom{n}{j} \sum_{\substack{2r+s=k-1 \\ r,s \geq 0}} (a_{2r}(j) + a_{2r+1}(j)) a_s(n-j). \quad (8)$$

*Proof.* We can choose a permutation  $w = a_1 \cdots a_{n+1} \in \mathfrak{S}_{n+1}$  such that  $\text{as}(w) = k$  as follows. First choose  $0 \leq j \leq n$  such that  $a_{j+1} = n + 1$ . Then choose in  $\binom{n}{j}$  ways the set  $\{a_1, \dots, a_j\}$ . For  $s \geq 0$  we can choose in  $a_s(n-j)$  ways a permutation  $w' = a_{j+2} \cdots a_{n+1}$  satisfying  $\text{as}(w') = s$ . Next we choose a permutation  $w'' = a_1 \cdots a_j$  such that the longest *even* length of an alternating subsequence of  $w''$  is  $2r = k - 1 - s$ . We can choose  $w''$  to satisfy either  $\text{as}(w'') = 2r$  or  $\text{as}(w'') = 2r + 1$ . The concatenation  $w = w''(n+1)w' \in \mathfrak{S}_{n+1}$  will then satisfy  $\text{as}(w) = k$ , and conversely all such  $w$  arise in this way. Hence equation (8) follows.  $\square$

Now write

$$F_k(x) = \sum_{n \geq 0} a_k(n) \frac{x^n}{n!}.$$

For instance,  $F_0(x) = 1$  and  $F_1(x) = e^x - 1$ . Multiplying (8) by  $x^n/n!$  and summing on  $n \geq 0$  gives

$$F'_k(x) = \sum_{2r+s=k-1} (F_{2r}(x) + F_{2r+1}(x)) F'_s(x). \quad (9)$$

Note that

$$A(x, t) = \sum_{k \geq 0} F_k(x) t^k,$$

where  $A(x, t)$  is defined by (5). Since  $k - 1 - s$  is even in (9), we need to work with the even part  $A_e(x, t)$  and odd part  $A_o(x, t)$  of  $A(x, t)$ , defined by

$$\begin{aligned} A_e(x, t) &= \sum_{k \geq 0} F_{2k}(x) t^{2k} \\ &= \frac{1}{2}(A(x, t) + A(x, -t)) \\ A_o(x, t) &= \sum_{k \geq 0} F_{2k+1}(x) t^{2k+1} \\ &= \frac{1}{2}(A(x, t) - A(x, -t)). \end{aligned} \tag{10}$$

Multiply equation (9) by  $t^k$  and sum on  $k \geq 0$ . We obtain

$$\frac{\partial A(x, t)}{\partial x} = t A_e(x, t) A(x, t) + A_o(x, t) A(x, t). \tag{11}$$

Substituting  $-t$  for  $t$  yields

$$\frac{\partial A(x, -t)}{\partial x} = -t A_e(x, t) A(x, -t) - A_o(x, t) A(x, -t). \tag{12}$$

Adding and subtracting equations (11) and (12) gives the following system of differential equations for  $A_e = A_e(x, t)$  and  $A_o = A_o(x, t)$ :

$$\frac{\partial A_e}{\partial x} = t A_e A_o + A_o^2 \tag{13}$$

$$\frac{\partial A_o}{\partial x} = t A_e^2 + A_e A_o. \tag{14}$$

Thus we need to solve this system of equations in order to find  $A(x, t) = A_e(x, t) + A_o(x, t)$ .

**Theorem 2.3.** *We have*

$$B(x, t) = \frac{1 + \rho + 2te^{\rho x} + (1 - \rho)e^{2\rho x}}{1 + \rho - t^2 + (1 - \rho - t^2)e^{2\rho x}} \quad (15)$$

$$A(x, t) = (1 - t)B(x, t) \quad (16)$$

$$= (1 - t) \frac{1 + \rho + 2te^{\rho x} + (1 - \rho)e^{2\rho x}}{1 + \rho - t^2 + (1 - \rho - t^2)e^{2\rho x}}, \quad (17)$$

where  $\rho = \sqrt{1 - t^2}$ .

*Proof.* We can simply verify that the stated expression (17) for  $A(x, t)$  satisfies (13) and (14) with the initial condition  $A(0, t) = 1$ , a routine computation (especially with the use of a computer). The relationship (16) between  $A(x, t)$  and  $B(x, t)$  is then an immediate consequence of (4), which is equivalent to  $a_k(n) = b_k(n) - b_k(n - 1)$ .

It might be of interest, however, to explain how the formula (17) for  $A(x, t)$  can be derived if the answer is not known in advance. If we divide equation (13) by (14), then we obtain

$$\frac{\partial A_e / \partial x}{\partial A_o / \partial x} = \frac{A_o}{A_e}.$$

Hence  $\frac{\partial}{\partial x}(A_e^2 - A_o^2) = 0$ , so  $A_e^2 - A_o^2$  is independent of  $x$ . This observation suggests computing the generating function in  $t$  for  $A_e^2 - A_o^2$ , which the computer shows is equal to  $1 + O(t^N)$  for a large value of  $N$ . Assuming then that  $A_e^2 - A_o^2 = 1$  (or even proving it combinatorially), we can substitute  $\sqrt{1 - A_e^2}$  for  $A_o$  in (13) to obtain

$$\frac{\partial A_e}{\partial x} = tA_e\sqrt{A_e^2 - 1} + A_e^2 - 1,$$

a single differential equation for  $A_e$ . This equation can routinely be solved by separation of variables (though some care must be taken to choose the correct branch of the resulting integral, including the correct sign of  $\sqrt{A_e^2 - 1}$ ); we will spare the reader the details. A similar argument yields  $A_o$ , so we obtain  $A = A_e + A_o$ .  $\square$

NOTE. Ira Gessel has pointed out the following simplified expression for  $B(x, t)$ :

$$B(x, t) = \frac{2/\rho}{1 - \frac{1-\rho}{t}e^{\rho x}} - \frac{1}{\sqrt{1-t^2}}. \quad (18)$$

### 3 Consequences.

A number of corollaries follow from Theorem 2.3. The first is the explicit expressions for  $a_k(n)$  and  $b_k(n)$  stated in the introduction. I am grateful to Ira Gessel for providing the proof given below.

**Corollary 3.1.** *For all  $k, n \geq 1$  we have*

$$b_k(n) = \frac{1}{2^{k-1}} \sum_{\substack{r+2s \leq k \\ r \equiv k \pmod{2}}} (-2)^s \binom{k-s}{(k+r)/2} \binom{n}{s} r^n \quad (19)$$

$$a_k(n) = b_k(n) - b_{k-1}(n). \quad (20)$$

*Proof.* Define  $b'_k(n)$  to be the right-hand side of (19), and set

$$B'(x, t) = \sum_{k, n \geq 0} b'_k(n) t^k \frac{x^n}{n!}.$$

Set  $n = s + m$  and  $k = r + 2s + 2l$ , so

$$\begin{aligned} B'(x, t) &= \sum_{r, s, l, m} (-1)^s 2^{1-r-s-2l} \binom{r+s+2l}{r+s+l} \binom{s+m}{s} r^{s+m} t^{r+2s+2l} \frac{x^{s+m}}{(s+m)!} \\ &= 2 \sum_{r, s \geq 0} \left(\frac{t}{2}\right)^r \frac{(-rt^2x/2)^s}{s!} \left[ \sum_l \binom{r+s+2l}{l} \left(\frac{t^2}{4}\right)^l \right] \left[ \sum_m \frac{(rx)^m}{m!} \right]. \end{aligned}$$

The sum on  $m$  is  $e^{rx}$ . Using the formula

$$\sum_k \binom{2k+a}{k} u^k = \frac{C(u)^a}{\sqrt{1-4u}},$$



where

$$C(u) = \sum_{n \geq 0} C_n u^n = \frac{1 - \sqrt{1 - 4x}}{2x},$$

the generating function for the Catalan numbers  $C_n = \frac{1}{n+1} \binom{2n}{n}$ , we find that the sum on  $l$  is

$$\frac{C(t^2/4)^{r+s}}{\sqrt{1-t^2}} = \frac{1}{\rho} \left( \frac{2-2\rho}{t^2} \right)^{r+s}.$$

Thus

$$\begin{aligned} B'(x, t) &= \frac{2}{\rho} \sum_{r, s \geq 0} \left( \frac{t}{2} \right)^r \frac{(-rt^2x/2)^s}{s!} e^{rx} \left( \frac{2-2\rho}{t^2} \right)^{r+s} \\ &= \frac{2}{\rho} \sum_r \left( \frac{1-\rho}{t} e^x \right)^r \sum_s \frac{(-r(1-\rho)x)^s}{s!} \\ &= \frac{2}{\rho} \sum_r \left( \frac{1-\rho}{t} e^x \right)^r e^{-r(1-\rho)x} \\ &= \frac{2}{\rho} \frac{1}{1 - \frac{1-\rho}{t} e^{\rho x}}, \end{aligned}$$

and the proof of (19) follows from (18). Equation (20) is then an immediate consequence of (4).  $\square$

By Corollary 3.1, when  $k$  is fixed  $b_k(n)$  is a linear combination of  $k^n, (k-2)^n, (k-4)^n, \dots$  with coefficients that are polynomials in  $n$ . For  $k \leq 6$  we have

$$\begin{aligned} b_2(n) &= 2^{n-1} \\ b_3(n) &= \frac{1}{4}(3^n - 2n + 3) \\ b_4(n) &= \frac{1}{8}(4^n - 2(n-2)2^n) \\ b_5(n) &= \frac{1}{16}(5^n - (2n-5)3^n + 2(n^2 - 5n + 5)) \\ b_6(n) &= \frac{1}{32}(6^n - 2(n-3)4^n + (2n^2 - 12n + 15)2^n). \end{aligned}$$

As a further application of Theorem 2.3 we can obtain the factorial moment generating function

$$F(x, t) = \sum_{s, n \geq 0} \nu_j(n) x^n \frac{t^j}{j!},$$

where

$$\nu_j(n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} (\text{as}(w))_j = \frac{1}{n!} \sum_k a_k(n) (k)_j.$$

and

$$(h)_j = h(h-1) \cdots (h-j+1).$$

Namely, we have

$$\begin{aligned} \left. \frac{\partial^j A(x, t)}{\partial t^j} \right|_{t=1} &= \sum_{n \geq 0} \frac{1}{n!} \sum_{k \geq 0} a_k(n) (k)_j x^n \\ &= \sum_{n \geq 0} \nu_j(n) x^n. \end{aligned}$$

On the other hand, by Taylor's theorem we have

$$A(x, t) = \sum_{j \geq 0} \left. \frac{\partial^j A(x, t)}{\partial t^j} \right|_{t=1} \frac{(t-1)^j}{j!}.$$

It follows that

$$F(x, t) = A(x, t+1). \tag{21}$$

(Note that it is not at all a priori obvious from the form of  $A(x, t+1)$  obtained by substituting  $t+1$  for  $t$  in (17) that it even has a Taylor series expansion at  $t=0$ .) From equations (17) and (21) it is easy to compute (using a computer) the generating functions

$$M_j(x) = \sum_{n \geq 0} \nu_j(n) x^n$$

for small  $j$ . For  $1 \leq j \leq 4$  we get

$$\begin{aligned}
M_1(x) &= \frac{6x - 3x^2 + x^3}{6(1-x)^2} \\
M_2(x) &= \frac{90x^2 - 15x^4 + 6x^5 - x^6}{90(1-x)^3} \\
M_3(x) &= \frac{2520x^3 - 315x^4 + 189x^5 - 231x^6 + 93x^7 - 18x^8 + 2x^9}{1260(1-x)^4} \\
M_4(x) &= \frac{N_4(x)}{9450(1-x)^5},
\end{aligned}$$

where

$$\begin{aligned}
N_4(x) &= 47250x^4 - 3780x^6 + 2880x^7 - 2385x^8 + 1060x^9 - 258x^{10} \\
&\quad + 36x^{11} - 3x^{12}.
\end{aligned}$$

It is not difficult to see that in general  $M_j(x)$  is a rational function of  $x$  with denominator  $(1-x)^{j+1}$ . It follows from standard properties of rational generating functions [15, §4.3] that for fixed  $j$  we have that  $\nu_j(n)$  is a polynomial in  $n$  of degree  $j$  for  $n$  sufficiently large. In particular, we have

$$\begin{aligned}
\nu_1(n) &= \frac{4n+1}{6}, \quad n \geq 2 \\
\nu_2(n) &= \frac{40n^2 - 24n - 19}{90}, \quad n \geq 4 \\
\nu_3(n) &= \frac{1120n^3 - 2856n^2 + 440n + 1581}{3780}, \quad n \geq 6.
\end{aligned} \tag{22}$$

Note in particular that  $\nu_1(n)$  is just the expectation (mean) of  $as_n$ . The simple formula  $(4n+1)/6$  for this quantity should be contrasted with the situation for the length  $is_n(w)$  of the longest increasing subsequence of  $w \in \mathfrak{S}_n$ , where even the asymptotic formula  $E(n) \sim 2\sqrt{n}$  for the expectation is a highly nontrivial result [17, §3]. A simple proof of (22) follows from (27) and an argument of Knuth [10, Exer. 5.1.3.15].

From the formulas for  $\nu_1(n)$  and  $\nu_2(n)$  we easily compute the variance  $\text{var}(\text{as}_n)$  of  $\text{as}_n$ , namely,

$$\text{var}(\text{as}_n) = \nu_2(n) + \nu_1(n) - \nu_1(n)^2 = \frac{32n - 13}{180}, \quad n \geq 4. \quad (23)$$

We now consider a further application of Theorem 2.3. Let

$$T_n(t) = \sum_{k=0}^n a_k(n)t^k. \quad (24)$$

For instance,

$$T_1(t) = t$$

$$T_2(t) = t + t^2$$

$$T_3(t) = t + 3t^2 + 2t^3$$

$$T_4(t) = t + 7t^2 + 11t^3 + 5t^4$$

$$T_5(t) = t + 15t^2 + 43t^3 + 45t^4 + 16t^5$$

$$T_6(t) = t + 31t^2 + 148t^3 + 268t^4 + 211t^5 + 61t^6$$

$$T_7(t) = t + 63t^2 + 480t^3 + 1344t^4 + 1767t^5 + 1113t^6 + 272t^7.$$

**Corollary 3.2.** *The polynomial  $T_n(t)$  is divisible by  $(1+t)^{\lfloor n/2 \rfloor}$ . Moreover, if  $U_n(t) = T_n(t)/(1+t)^{\lfloor n/2 \rfloor}$ , then*

$$U_{2n}(-1) = -U_{2n+1}(-1) = \frac{(-1)^n E_{2n+1}}{2^n},$$

where  $E_{2n+1}$  denotes a tangent number.

*Proof.* Let  $A_e(x, t)$  and  $A_o(x, t)$  be the even and odd parts of  $A(x, t)$  as in equation (10). By the definition of  $A_e(x)$  we have

$$A_e(x/\sqrt{1+t}, t) = \sum_{n \geq 0} \frac{T_{2n}(t)}{(1+t)^n} \frac{x^{2n}}{(2n)!}.$$

With the help of the computer we compute that

$$\begin{aligned} \lim_{t \rightarrow -1} A_e(x/\sqrt{1+t}, t) &= \text{sech}^2 \frac{x}{\sqrt{2}} \\ &= \sum_{n \geq 0} \frac{(-1)^n E_{2n+1}}{2^n} \frac{x^{2n}}{(2n)!}. \end{aligned}$$

Hence the desired result is true for  $T_{2n}(t)$ . Similarly,

$$\begin{aligned} \lim_{t \rightarrow -1} \sqrt{1+t} A_o(x/\sqrt{1+t}, t) &= -\sqrt{2} \tanh \frac{x}{\sqrt{2}} \\ &= -\sum_{n \geq 0} \frac{(-1)^n E_{2n+1}}{2^n} \frac{x^{2n+1}}{(2n+1)!}, \end{aligned}$$

proving the result for  $T_{2n+1}(t)$ .  $\square$

By Corollary 3.2 we have  $T_n(-1) = 0$  for  $n \geq 2$ . In other words, for  $n \geq 2$  we have

$$\#\{w \in \mathfrak{S}_n : \text{as}_n(w) \text{ even}\} = \#\{w \in \mathfrak{S}_n : \text{as}_n(w) \text{ odd}\} = \frac{n!}{2}.$$

A simple combinatorial proof of this fact follows from switching the last two elements of  $w$ ; it is easy to see that this operation either increases or decreases  $\text{as}_n(w)$  by 1, as first pointed out by M. Bóna and P. Pylyavskyy. More generally, a combinatorial proof of Corollary (3.2) is a consequence of equation (27) below and an argument of Bóna [6, Lemma 1.40].

The formulas (22) and (23) for the mean and variance of  $\text{as}_n$  suggest in analogy with (2) that  $\text{as}_n$  will have a limiting distribution  $K(t)$  defined by

$$K(t) = \lim_{n \rightarrow \infty} \text{Prob} \left( \frac{\text{as}_n(w) - 2n/3}{\sqrt{n}} \leq t \right),$$

for all  $t \in \mathbb{R}$ , where  $w$  is chosen uniformly from  $\mathfrak{S}_n$ . Indeed, we have that  $K(t)$  is a Gaussian distribution with variance  $8/45$ :

$$K(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{t\sqrt{45}/4} e^{-s^2} ds. \quad (25)$$

It was pointed out by Pemantle (private communication) that equation (25) is a consequence of the result [13, Thms. 3.1, 3.3, or 3.5] and possibly also [5]. An independent proof was also given by Widom [19], and in the next section we explain an additional method of proof.

## 4 Relationship to alternating runs.

A *run* of a permutation  $w = w_1 \cdots w_n \in \mathfrak{S}_n$  is a maximal factor (subsequence of consecutive elements) which is increasing. An *alternating run* is a maximal factor that is increasing or decreasing. (Perhaps “birun” would be a better term.) For instance, the permutation 64283157 has four alternating runs, viz., 642, 28, 831, and 157. Let  $g_k(n)$  be the number of permutations  $w \in \mathfrak{S}_n$  with  $k$  alternating runs. It is easy to see, as pointed out by Bóna [7], that

$$a_k(n) = \frac{1}{2}(g_{k-1}(n) + g_k(n)), \quad n \geq 2. \quad (26)$$

If we define  $G_n(t) = \sum_k g_k(n)t^k$ , then equation (26) is equivalent to the formula

$$T_n(t) = \frac{1}{2}(1+t)G_n(t), \quad (27)$$

where  $T_n(t)$  is defined by (24).

Research on the numbers  $g_k(n)$  go back to the nineteenth century; for references see Bona [6, §1.2] and Knuth [10, Exer. 5.1.3.15–16]. In particular, let  $A_n(t)$  denote the  $n$ th *Eulerian polynomial*, i.e.,

$$A_n(t) = \sum_{w \in \mathfrak{S}_n} t^{1+\text{des}(w)},$$

where  $\text{des}(w)$  denotes the number of descents of  $w$  (the size of the descent set defined in equation (28)). It was shown by David and Barton [8, pp. 157–162] and stated more concisely by Knuth [10, p. 605] that

$$G_n(t) = \left(\frac{1+t}{2}\right)^{n-1} (1+w)^{n+1} A_n\left(\frac{1-w}{1+w}\right), \quad n \geq 2,$$

where  $w = \sqrt{\frac{1-t}{1+t}}$ . Theorem 2.3 is then a straightforward consequence of the well-known generating function (e.g., [6, Thm. 1.7])

$$\sum_{n \geq 0} A_n(t) \frac{x^n}{n!} = \frac{1-t}{1-te^{(1-t)x}}.$$

It is also well-known (e.g., [6, Thm. 1.10]) that the Eulerian polynomial  $A_n(t)$  has only real zeros, and that the zeros of  $A_n(t)$  and  $A_{n+1}(t)$  interlace. From this fact Wilf [20] showed that the polynomials  $G_n(t)$  have (interlacing) real zeros, and hence by (27) the polynomials  $T_n(t)$  also have real zeros. It is then a consequence of standard results (e.g., [4, Thm. 2]) that the numbers  $a_k(n)$  for fixed  $n$  are asymptotically normal as  $n \rightarrow \infty$ , yielding another proof of (25).

## 5 Open problems.

In this section we mention three directions of possible generalization of our work above.

1. Let  $\text{is}(m, w)$  denote the length of the longest subsequence of  $w \in \mathfrak{S}_n$  that is a union of  $m$  increasing subsequences, so  $\text{is}(w) = \text{is}(1, w)$ . The numbers  $\text{is}(m, w)$  have many interesting properties, summarized in [17, §4]. Can anything be said about the analogue for alternating sequences, i.e., the length  $\text{as}(m, w)$  of the longest subsequence of  $w$  that is a union of  $m$  alternating subsequences? This question can also be formulated in terms of the lengths of the alternating runs of  $w$ .
2. Can the results for increasing subsequences and alternating subsequences be generalized to other “patterns”? More specifically, let  $\sigma$  be a (finite) word in the letters  $U$  and  $D$ , e.g.,  $\sigma = UUDUD$ . Let  $\sigma^\infty$  denote the infinite word  $\sigma\sigma\sigma\cdots$ , e.g.,

$$(UUD)^\infty = UUDUUDUUD \dots$$

For this example, we have for instance that  $UUDUUDU$  is a prefix of  $\sigma^\infty$  of length 7.

Let  $\tau = a_1a_2 \cdots a_{m-1}$  be a word of length  $m - 1$  in the letters  $U$  and  $D$ . A sequence  $v = v_1v_2 \cdots v_m$  of integers is said to have *descent word*  $\tau$  if  $v_i > v_{i+1}$  whenever  $a_i = D$ , and  $v_i < v_{i+1}$  whenever  $a_i = U$ . Thus  $v$  is increasing if and only if

$\tau = U^{m-1}$ , and  $v$  is alternating if and only if  $\tau = (DU)^{j-1}$  or  $\tau = (DU)^{j-1}D$  depending on whether  $m = 2j - 1$  or  $m = 2j$ .

Now let  $w \in \mathfrak{S}_n$  and define  $\text{len}_\sigma(w)$  to be the length of longest subsequence of  $w$  whose descent word is a prefix of  $\sigma^\infty$ . Thus  $\text{len}_U(w) = \text{is}_n(w)$  and  $\text{len}_{DU}(w) = \text{as}_n(w)$ . What can be said in general about  $\text{len}_\sigma(w)$ ? In particular, let

$$E_\sigma(n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \text{len}_\sigma(w),$$

the expectation of  $\text{len}_\sigma(w)$  for  $w \in \mathfrak{S}_n$ . Note that  $E_U(n) \sim 2\sqrt{n}$  by (1), and  $E_{DU}(n) \sim 2n/3$  by (7). Is it true that for any  $\sigma$  we have  $E_\sigma(n) \sim \alpha n^c$  for some  $\alpha, c > 0$ ? Or at least that for some  $c > 0$  (depending on  $\sigma$ ) we have

$$\lim_{n \rightarrow \infty} \frac{\log E_\sigma(n)}{\log n} = c,$$

in which case can we determine  $c$  explicitly?

3. The *descent set*  $D(w)$  of a permutation  $w = w_1 \cdots w_n$  is defined by

$$D(w) = \{i : w_i > w_{i+1}\} \subseteq [n - 1], \quad (28)$$

where  $[n - 1] = \{1, 2, \dots, n - 1\}$ . Thus  $w$  is alternating if and only if  $D(w) = \{1, 3, 5, \dots\} \cap [n - 1]$ . Let  $S \subseteq [k - 1]$ . What can be said about the number  $b_{k,S}(n)$  of permutations  $w \in \mathfrak{S}_n$  that avoid all  $v \in \mathfrak{S}_k$  satisfying  $D(v) = S$ ? In particular, what is the value  $L_{k,S} = \lim_{n \rightarrow \infty} b_{k,S}(n)^{1/n}$ ? (It follows from [2] and [12], generalized in an obvious way, that this limit exists and is finite.) For instance, if  $S = \emptyset$  or  $S = [k - 1]$ , then it follows from [14] that  $L_{k,S} = (k - 1)^2$ . On the other hand, if  $S = \{1, 3, 5, \dots\} \cap [k - 1]$  then it follows from (19) that  $L_{k,S} = k - 1$ .



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